Ex: Compute $\int_0^5 12x^2y^3\,dx$ and $\int_0^1 12x^2y^3\,dy$, and the associated indefinite integrals.

Sol:
First $\int 12x^2y^3\,dx = 4x^3y^3 + g(y)$

notice that this is a function of y since "x sees y as a constant"

$$\int_0^5 12x^2y^3\,dx = 4x^3y^3\Big|_0^5 = 4\cdot 5^3\cdot y^3 - 0 = 500y^3$$

$$\int 12x^2y^3\,dy = 3x^2y^4 + h(x)$$

$$\int_0^1 12x^2y^3\,dy = 3x^2y^4\Big|_0^1 = 3x^2\cdot 1^4 - 0 = 3x^2$$

$\diamond$

Lecture 13

Ex: Compute:

ⓐ $\int_0^2\int_0^4 y^3e^{2x}\,dy\,dx$ ⓑ $\int_0^4\int_0^2 y^3e^{2x}\,dx\,dy$

Sol: ⓐ $\int_0^2\int_0^4 y^3e^{2x}\,dy\,dx = \int_0^2 \frac{1}{4}y^4e^{2x}\Big|_0^4\,dx$

$= \int_0^2 64e^{2x}\,dx = 32e^{2x}\Big|_0^2 = 32(e^4-1)$

ⓑ $\int_0^4\int_0^2 y^3e^{2x}\,dx\,dy = \int_0^4 \frac{1}{2}y^3e^{2x}\Big|_0^2\,dy$

$= \int_0^4 \frac{1}{2}(e^4-1)y^3\,dy = \frac{1}{8}(e^4-1)y^4\Big|_0^4 = \frac{1}{8}(e^4-1)\cdot 256$

$= 32(e^4-1)$.

$\diamond$

These integrals being equal is no coincidence:

Fubini's Theorem: If f is continuous on $R=[a,b]\times[c,d]$,

then: $$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

(There are more general conditions f could satisfy: f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the integrals exist.)

Ex: Compute $\iint_R ye^{-xy}\,dA$ where $R=[0,2]\times[0,3]$.

Sol: Since this looks annoying to integrate y first in, let's integrate w.r.t. x first. Then:

$$\iint_R ye^{-xy}\,dA = \int_0^3 \int_0^2 ye^{-xy}\,dx = \int_0^3 y\left(\frac{-1}{y}e^{-xy}\right)\Big|_0^2\,dy$$

$$= \int_0^3 \left[-e^{-2y} - (-1)\right] dy = \int_0^3 (1-e^{-2y})\,dy$$

$$= \left(y+\frac{1}{2}e^{-2y}\right)\Big|_0^3 = \left(3+\frac{1}{2}e^{-6}\right) - \left(0+\frac{1}{2}\right) = \frac{5}{2}+\frac{1}{2}e^{-6}$$

◇

Let's end with a volume example.

Ex: Find the volume of the solid bounded by the surface $z = 1 + e^x \sin y$ and the planes $x = 1$, $x = -1$, $y = 0$, $y = \pi$, and $z = 0$.

Sol: Since $e^x > 0$ and $\sin y \geq 0$ on $0 \leq y \leq \pi$, we have $z \geq 0$. The base of this solid is $R = [-1, 1] \times [0, \pi]$ and its height is z. Thus,

$$\text{Vol} = \iint_R (1 + e^x \sin y)\, dA = \int_{-1}^{1} \int_0^{\pi} (1 + e^x \sin y)\, dy\, dx$$

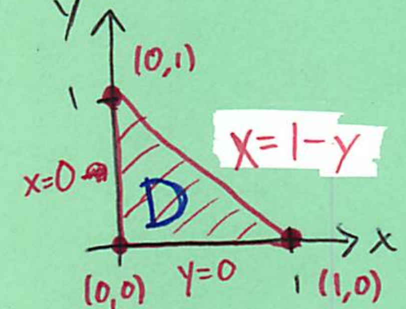
$$= \int_{-1}^{1} (y - e^x \cos y)\Big|_0^{\pi} dx = \int_{-1}^{1} \left[(\pi + e^x) - (-e^x) \right] dx$$

$$= \int_{-1}^{1} (2e^x + \pi)\, dx = (2e^x + \pi x)\Big|_{-1}^{1} = (2e + \pi) - (2e^{-1} - \pi)$$

$$= 2e - 2e^{-1} + 2\pi. \qquad \diamond$$

15.3 - Double Integrals over General Regions

Let's consider the problem of integrating the function $z=5$ over the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$.



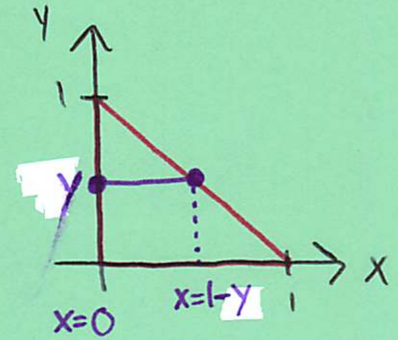
So, we're computing:

$$\iint_D 5\,dA$$

where D is the triangular region.

How do we compute this? Slices.

Recall that to perform a double integral, we compute the inner integral first by holding one variable constant and integrating w.r.t. the other. Let's say we integrate with respect to x first. This means we fix y and integrate from the smallest x-value to the largest at this y-value:



"horizontal slices"

So, the bounds on the x-integral are $0 \le x \le 1-y$. Now, all we have to do is add up over all the possible y-values. So,

$$\int_0^1 \int_0^{1-y} 5\,dx\,dy = \int_0^1 5x\Big|_0^{1-y} dy = \int_0^1 (5-5y)\,dy$$

$$= \left(5y - \frac{5}{2}y^2\right)\Big|_0^1 = 5 - \frac{5}{2} = \frac{5}{2}$$

out of a double integral

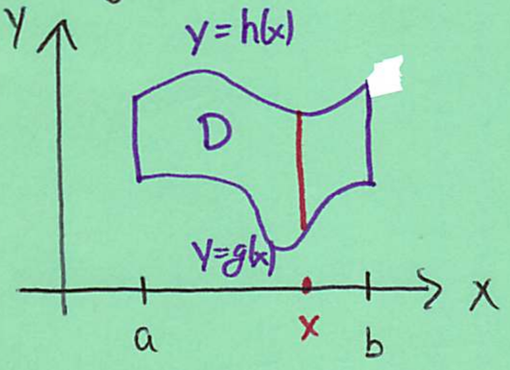
Comments:

1) The outside integral should NEVER have a variable in it!

2) ALWAYS sketch the region of integration. It really helps when setting up bounds.

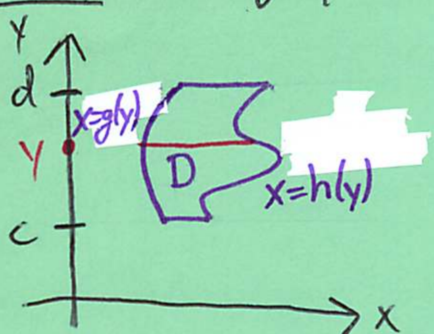
3) To find bounds, first sketch the region, then decide which way to take slices:

• vertical (holding x constant first): look from bottom to top:



$$\iint_D f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

• horizontal (holding y constant first): look from left to right:



$$\iint_D f(x,y)\,dA = \int_c^d \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$$

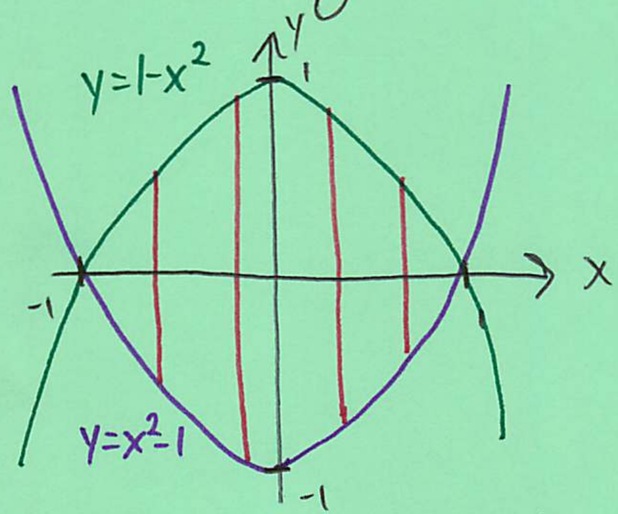
Area of D $= A(D) = \iint_D dA = \int_{-1}^{2} \int_{y^2-1}^{y+1} dA$

You also need to be able to read the region of integration off of a double integral:

4) Sometimes integrating one way is easier than another.

Ex: Compute $\iint_D x\,dA$ where D is the region bounded by the parabolas $y = 1-x^2$ & $y = x^2-1$.

Sol: Step 1: sketch the region!



Notice that vertical slices work better here since to do horizontal would require splitting the integral into two pieces (also, the bounds wouldn't be as nice!).

Step 2: Set up the integral:

$$\iint_D x\,dA = \int_{-1}^{1}\int_{x^2-1}^{1-x^2} x\,dy\,dx = \int_{-1}^{1} xy\Big|_{x^2-1}^{1-x^2} dx$$

$$= \int_{-1}^{1} x\left[(1-x^2)-(x^2-1)\right]dx = \int_{-1}^{1}(2x-2x^3)\,dx$$

$$= \left(x^2 - \tfrac{1}{2}x^4\right)\Big|_{-1}^{1} = \left(1-\tfrac{1}{2}\right) - \left(1-\tfrac{1}{2}\right) = 0$$

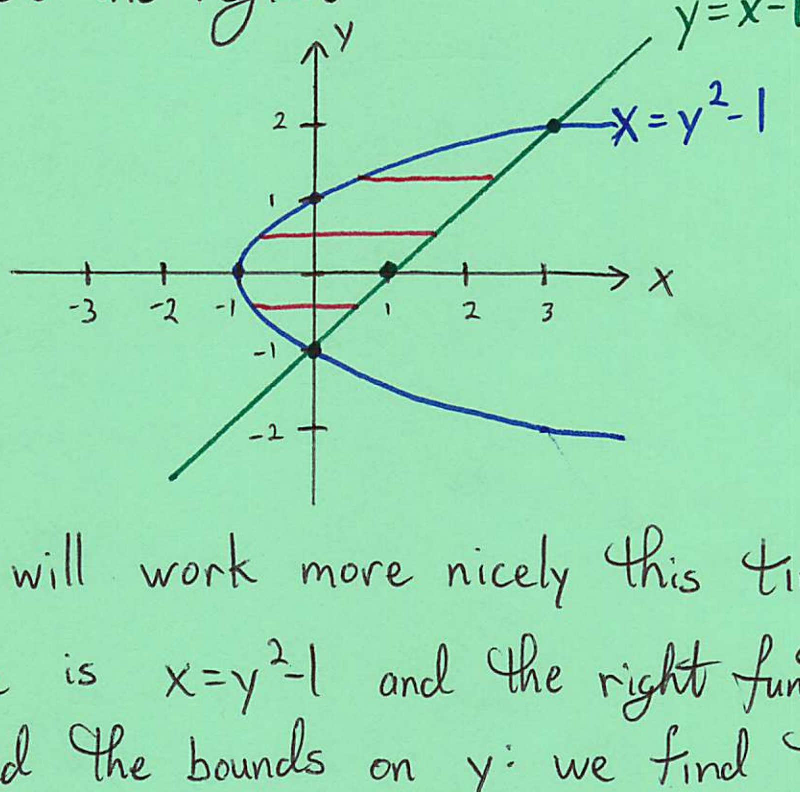
From now on, we'll focus more on setting up integrals.

Ex: Set up the integral to find the area of the region bounded by $x=y^2-1$ and $y=x-1$.

Notice that the volume of an object with height 1 is equal to the area of its base (ignoring units, of course).

So, area of $D = A(D) = \iint_D 1\,dA = \iint_D dA$.

Sol: First, sketch the region:



Horizontal slices will work more nicely this time.

The left function is $x=y^2-1$ and the right function is $x=y+1$.

Now, we just need the bounds on y: we find the max and min values of y: plugging $x=y^2-1$ into $y=x-1$ gives:

$$y=y^2-2 \Rightarrow y^2-y-2=(y-2)(y+1)=0 \Rightarrow y=-1,2. \text{ Thus:}$$

$$\text{Area of } D = A(D) = \iint_D dA = \int_{-1}^{2}\int_{y^2-1}^{y+1} dA$$

◇

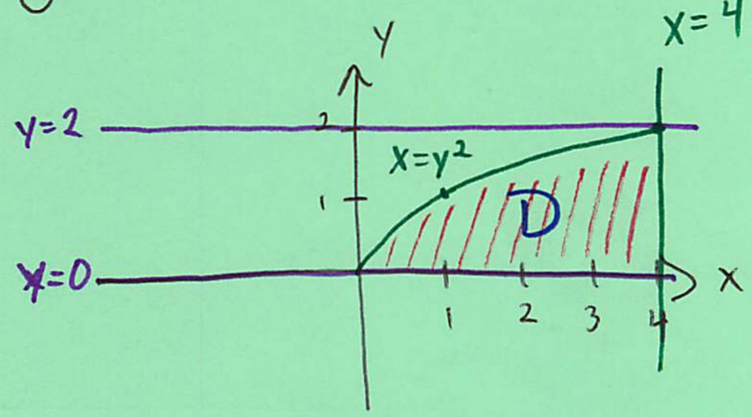
You also need to be able to read the region of integration off of a double integral:

Ex: Compute $\int_0^2 \int_{y^2}^{4} y\cos(x^2)\,dx\,dy$.

Sol: As it stands, we cannot compute it since we cannot compute $\int \cos(x^2)\,dx$ ... However, we can try to switch the order in which we integrate to $dy\,dx$.

To do this, sketch the region:

- the outer integral says $0 \le y \le 2$
- x goes between y^2 and 4 for any fixed y, so $y^2 \le x \le 4$.



We only draw the pieces between the purple lines.

So, rewriting the integral we have:

$$\int_0^2 \int_{y^2}^{4} y\cos(x^2)\,dx\,dy = \int_0^4 \int_0^{\sqrt{x}} y\cos(x^2)\,dy\,dx$$

$$= \int_0^4 \frac{1}{2}y^2\cos(x^2)\Big|_0^{\sqrt{x}}\,dx = \int_0^4 \frac{1}{2}x\cos(x^2)\,dx$$

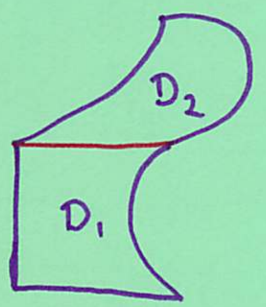
$$= \int_0^{16} \frac{1}{4}\cos u\,du = \frac{1}{4}\sin u\Big|_0^{16} = \frac{1}{4}\sin 16.$$

$\diamond$

Last comment: If $D = D_1 \cup D_2$, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

e.g.:

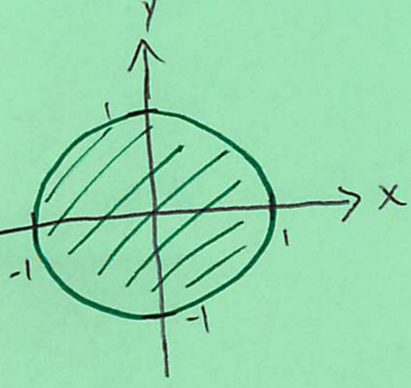
$D =$  $= D_1 \cup D_2$

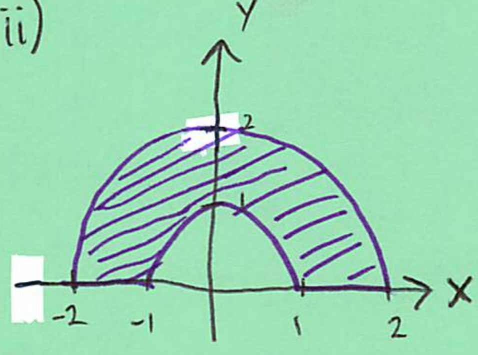
Sometimes there's no choice but to split the integral in two...

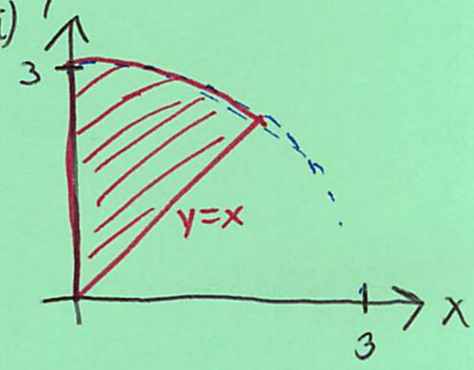
15.4 - Double Integrals in Polar Coordinates.

Consider the integral: $\iint_D e^{x^2+y^2}\,dA$, where D is the unit disk. How can we compute it? The answer is polar coordinates. Let's practice describing regions in polar coordinates.

Ex: Describe the following regions in polar coordinates:

i) 

ii) 

iii) 

Sol: i) $D = \{(r,\theta) \mid r \le 1\} = \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

ii) $D = \{(r,\theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$

iii) $D = \{(r,\theta) \mid r \le 3, \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\}$

$\Diamond$

These types of regions are called polar rectangles since if you graph them in the r,θ-plane, they're rectangles. The most general polar rectangle is a sector of the form

$$D = \{(r,\theta) \mid r_1 \le r \le r_2, \theta_1 \le \theta \le \theta_2\}. \quad (0 \le \theta_2 - \theta_1 \le 2\pi)$$

The area of D is $(\Delta\theta = \theta_2 - \theta_1, \Delta r = r_2 - r_1, r^* = \frac{1}{2}(r_1 + r_2))$

$$A = \frac{1}{2} r_2^2 \Delta\theta - \frac{1}{2} r_1^2 \Delta\theta = \frac{1}{2}(r_2 + r_1)(r_2 - r_1)\Delta\theta$$

$$= r^* \Delta r \Delta\theta \rightsquigarrow dA = r\,dr\,d\theta$$

So, when changing from Cartesian to polar, we have:

$$\boxed{\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

(one can also start with the θ-integral).

Ex: Compute $\iint_D e^{x^2+y^2} dA$ where D is the unit disk.

Sol: D, in polar coordinates is, $\{(r,\theta) \mid r \le 1\}$. The integrand becomes e^{r^2} $(x^2+y^2 = r^2)$, so we get:

$$\iint_D e^{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta \overset{u=r^2}{=} \int_0^{2\pi}\int_0^1 \frac{1}{2}e^u\,du\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(e-1)\,d\theta = \pi(e-1).$$

◇

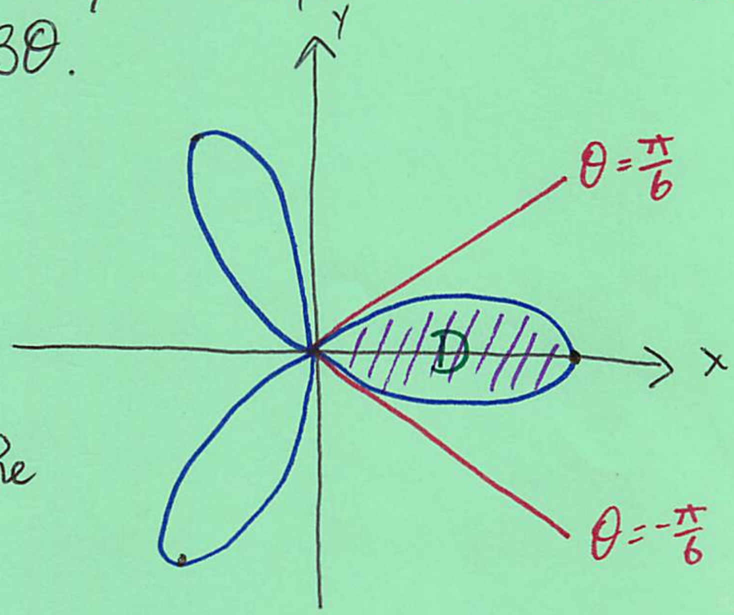
Lecture 14

Regions of integration need not be polar rectangles, naturally. Consider the following problem from Calc II:

Ex: Find the area enclosed by one petal of the rose

$$r = \cos 3\theta.$$

Sol: The rose looks like:



We know $\cos 3\theta = 0$ when $3\theta = \frac{\pi}{2} + n\pi \iff \theta = \frac{\pi}{6} + \frac{n\pi}{3}$.

Taking $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$, we get the indicated petal. So, to get the bounds on r, we fix a θ-value (a ray coming out of the origin) and find an "inner" and "outer" bound on r: in this case, $0 \le r \le \cos 3\theta$. So,

$$A(D) = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\cos^2 3\theta\,d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{4}(1+\cos 6\theta)\,d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{1}{4}\left[\left(\frac{\pi}{6} + \frac{1}{6}\sin\pi\right) - \left(\frac{-\pi}{6} + \frac{1}{6}\sin(-\pi)\right)\right]$$

$$= \frac{\pi}{12}$$

◇